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Two Alternatives for the Cubic Algorithm

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Two alternatives are proved that allow application of the cubic algorithm for finding approximate solutions of the general non-convex and non-robust mathematical programming problem.

1. INTRODUCTION

We consider two problems:

PROBLEM A. Find p^0 and X^0 such that

$$p^0 = \min f(x) \qquad \text{given} \quad X \subset R^n, \tag{1}$$

$$X^0 = \{x \mid f(x) = p^0, x \in X\}, \tag{2}$$

where X is a compact set which may be non-convex, non-connected, and non-robust (robust set Y is such that the closure of its interior coincides with its closure: cl int $Y =$ cl Y). For example, X may consist of a closed ball, several closed arcs (that may intersect each other and the ball) and several isolated points.

PROBLEM B. Find s^0 and K^0 such that

$$s^0 = \min_{x \in C} f(x), \qquad C \subset R^n, \tag{3}$$

$$K^0 = \{x \mid f(x) = s^0, x \in C\}, \tag{4}$$

where C is a closed cube such that

$$X \subset C \subset R^n. \tag{5}$$

HYPOTHESIS. *The cost function f: $R^n \to R$ is defined and Lipschitzian in C, that is,*

$$|f(x) - f(x')| \leqslant L \, \|x - x'\|, \qquad L = \text{const.} \quad x, x' \in C. \tag{6}$$

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Under this hypothesis, Problem B can be solved by the Cubic Algorithm, see [1], which provides the construction of the sequence of enclosed sets

$$C = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_j \supseteq \cdots \tag{7}$$

and corresponding comparison constants

$$s_0 \geqslant s_1 \geqslant \cdots \geqslant s_j \geqslant \cdots \tag{8}$$

such that

$$\lim_{j \to \infty} K_j = \bigcap_{j=0}^{\infty} K_j = K^0 \tag{9}$$

and

$$\lim_{j \to \infty} s_j = s^0. \tag{10}$$

Since $X \subset C$, so it is clear that $p^0 \geqslant s^0$. However, it is not evident in what relation the corresponding minimizing sets X^0 and K^0 may be.

2. The First Alternative

THEOREM 1. *Either*

$$X^0 \cap K^0 = \varnothing \tag{11}$$

or

$$X^0 \subseteq K^0. \tag{12}$$

Proof. Either $p^0 > s^0$, or $p^0 = s^0$. If $p^0 > s^0$, then (11) holds since otherwise there would be a minimizer $z^0 \in X^0 \cap K^0$ for which $f(z^0) = p^0$ due to $z^0 \in X^0$ and the same $f(z^0) = s^0$ due to $z^0 \in K^0$ yielding $p^0 = s^0$, in contradiction with the assumption $p^0 > s^0$.

If $p^0 = s^0$, then for every $x^0 \in X^0$ we have $f(x^0) = p^0 = s^0$ and, since $X^0 \subseteq X \subset C$, so by definition (4) every x^0 belongs to K^0 and inclusion (12) follows. ∎

The set K^0 is known as determined by the cubic algorithm. The set X^0 is unknown. However, the alternative can be formulated in the equivalent form making accent on the known sets X and K^0.

THEOREM 1a. *Either*

$$X \cap K^0 = \varnothing \tag{13}$$

or

$$X^0 = X \cap K^0. \tag{14}$$

Proof. If (13) holds, then (14) does not hold since X^0 is non-empty. On the other hand, if $X \cap K^0 \neq \varnothing$, then for every $x^0 \in X \cap K^0$ we have $f(x^0) = s^0$ and since by (1), (5) $f(x) \geqslant p^0 \geqslant s^0$ for all $x \in X$, so $x^0 = \arg\min_{x \in X} f(x)$ which implies $X \cap K^0 \subseteq X^0$ and $p^0 = s^0$. Now, from $p^0 = s^0$ we deduce, as above, the inclusion $X^0 \subseteq K^0$ and since also $X^0 \subseteq X$, so $X^0 \subseteq X \cap K^0$, and (14) follows. ∎

THEOREM 1b (the equivalence theorem). *The statements* (11) *and* (13), (12) *and* (14) *are pairwise equivalent.*

Proof. If (11) holds, then $p^0 > s^0$, so that $f(x) > s^0$ for all $x \in X$, that is,

$$f(x)\,|_{x \in X} > f(x)\,|_{x \in K^0} \tag{15}$$

which means (13). Vice versa, if (13) holds, then (11) is trivial since $X^0 \subset X$ (it also follows from (15), if we take the minimum over X on the left side).

If (12) holds, then $X \cap K^0 \neq \varnothing$ and (14) follows by Theorem 1a. If (14) holds, then (12) is trivial. ∎

Remark 1. The alternative is presented in application to the cubic algorithm which provides an iterative scheme to obtain K^0. However, the formulations of the theorems and their proofs do not use the procedure of the cubic algorithm, nor the properties of K_j, s_j. This means that, irrespective of the optimization method, the alternative holds for any two problems with common arbitrary (maybe, non-Lipschitzian) cost function and arbitrary (non-cubic) compact enclosed optimization sets $X \subset C$.

It is worth noting that compacity of X, C is *not* necessary. In the above theorems and proofs it is required only that the minimizing sets X^0 and K^0 be both non-empty.

EXAMPLE 1. As an immediate application of the first alternative, let us consider the problem of finding all real roots of a real n-degree polynomial $P_n(x, y)$ within a circle $x^2 + y^2 \leqslant r^2$. Take the cost function

$$f(x, y) = (x^2 + y^2)\, P_n^2(x, y).$$

Since $f(x, y) \geqslant 0$ and $f(0, 0) = 0$ so all global minimizers of $f(x, y)$ located within that circle, and only those minimizers, give the roots of $P_n(x, y)$

within the circle (with the exception, possibly, of the point $x=y=0$, if $P_n(0,0)\neq 0$).

However, the problem of minimizing $f(x, y)$ over the circle is not an easy one, given that a polynomial is usually not a convex function. To solve the problem, we take the square $C_r=\{x, y \mid |x|\leqslant r, |y|\leqslant r\}$ and apply the cubic algorithm to find all global minimizers $(x_i^0, y_i^0)\in C_r$, $i=1,..., k$, $1\leqslant k\leqslant n+1$, of which one, namely, $x_1^0=y_1^0=0$ follows from the construction of $f(x, y)$. This is an easy task, see [1], and then it remains to determine those (x_i^0, y_i^0) that belong to the circle, i.e., for which $x_i^{0^2}+y_i^{0^2}\leqslant r^2$. Indeed, the minimizer $(0, 0)$ belongs both to the circle and to the square, so (11) does not hold and, therefore, (12) affirms that all minima of $f(x, y)$ over the circle are contained in the set of its minima over the square C_r.

3. The Second Alternative

In general, minimization problems are solved not in one step but rather through certain iteration process. We consider here the set-monotonic processes which provide the sequences of the type (7), (8) yielding the solution as the limits (9) and (10).

Theorem 2. *Either after some* finite *number m of iterations it is established that*

$$K_m \cap X=\varnothing, \tag{16}$$

or otherwise

$$X^0\subseteq K^0. \tag{17}$$

Proof 2.1. Suppose (16) does not hold for any m, that is, $K_m\cap X\neq\varnothing$ for all m. Then there exists a sequence of points $z_m\in K_m$ such that $z_m\in X$ for all m. Since X is compact, there exists a subsequence z'_m that has a limit $z^0=\lim_{m\to\infty} z'_m$ which also belongs to X: $z^0\in X$. By construction (7), (9) we have also $z^0\in K^0$ meaning $K^0\cap X\neq\varnothing$, so that by the first alternative $X^0\subseteq K^0$. ∎

Proof 2.2. Suppose there exists m, such that (16) holds. Then, by virtue of the "enclosed" structure (7) we conclude that $K_j\cap X=\varnothing$ for all $j\geqslant m$ and, therefore, by (9) we have $K^0\cap X=\varnothing$, whence $K^0\cap X^0=\varnothing$. Now, by the first alternative, there may be only one different case, namely, $X^0\subseteq K^0$ in which case $K^0\cap X\neq\varnothing$ so that by (9), (7) we have $K_m\cap X\neq\varnothing$ for all m. ∎

Remark 2. The compacity of X and K was not explicitly used in the Proof 2.2 nor in the first alternative. If the minimizing sets X^0 and K^0 do exist (whether or not X and K are compact), then both alternatives still hold.

4. Application to the Approximate Solution of General Mathematical Programming Problems

Given Problem A, Eqs. (1) and (2), construct a closed cube C containing X as in (5). This is an easy task since we do not have to find the exactly circumscribed cube, it may be any closed cube around X. It is also convenient for further computations, if we make a linear transformation of the coordinate system, such that the cube C be a positive axes oriented cube with a peak in the origin. Then we can apply the same partition and grid generators as in the cubic algorithm [1].

Apply the cubic algorithm for Problem B and start obtaining sequences (7), (8), checking for each j whether or not there is a non-empty intersection

$$K_j \cap X \neq \varnothing, \qquad j = 1, 2, \ldots. \tag{18}$$

Obviously, $K_0 \cap X \neq \varnothing$ since $K_0 = C \supset X$, so that verification of (18) can be accomplished via separation algorithms constructed, e.g., on the basis of [2].

There may be two situations:

(1) The relation (18) holds for all $j = 1, 2, \ldots$. In this case by the second alternative we have $X^0 \subseteq K^0$, so that the cubic algorithm delivers the exact global minimum of $f(x)$ over X,

$$p^0 = s^0 \tag{19}$$

and the approximate minimizing set K^0 such that $X^0 \subseteq K^0$. If, in addition, we can compute the intersection $X \cap K^0$, then by Theorem 1b we can find the exact minimizing set for Problem A,

$$X^0 = X \cap K^0. \tag{20}$$

(2) The relation (18) does not hold for all $j = 1, 2, \ldots$ and there exists $j_0 = m$ such that

$$\bar{K}_m \cap \bar{X} \neq \varnothing, \tag{21}$$

$$\bar{K}_{m+1} \cap \bar{X} = \varnothing. \tag{22}$$

(From now on, $\bar{K}$ denotes the closure of K and K without bar denotes its interior (open set): $\bar{K} = \text{cl } K$, $K = \text{int } \bar{K}$, $\bar{K} - K = \partial K$, the boundary.) In this case, obviously, $\bar{K}_{m+1} \neq \bar{K}_m$, so that

$$\bar{Z}_{m+1} = \bar{K}_m - K_{m+1} \neq \varnothing. \tag{23}$$

Let $\bar{C}_i^{j+1} \subset \bar{K}_j$ be a closed subcube of $\bar{K}_j$ with the grid point $x_i^{j+1} \in \bar{C}_i^{j+1}$. We recall that the open C_i^{j+1} are disjoint and that the union of all closed $\bar{C}_i^{j+1}$ makes up the entire set $\bar{K}_j$. The procedure of the cubic algorithm is based [1] on the deletion operator given by the inequality

$$f(x_i^{j+1}) - s_j > r_{j+1}, \qquad j = 0, 1, \ldots. \tag{24}$$

Here $s_0 = f(x_0)$ where $x_0 \in \bar{K}_0$ is a point of grid in $\bar{K}_0$ (x_0 can be taken arbitrarily in $\bar{K}_0$, in [1] x_0 was taken at the origin: $x_0 = 0$). Subsequent comparison constants s_j ($j = 1, 2, \ldots$) are given by the extremal comparison constant generator

$$s_j = \min_{i \in I_j} f(x_i^j), \qquad j = 1, 2, \ldots, \tag{25}$$

where

$$I_{j+1} = \{ i \mid f(x_i^{j+1}) - s_j \leqslant r_{j+1}, \bar{C}_i^{j+1} \subset \bar{K}_j \}, \qquad j = 0, 1, \ldots \tag{26}$$

is obtained after deletion by (24).

Deletion constants r_j are

$$r_j = \frac{Lc\sqrt{n}}{N^j}, \qquad j = 1, 2, \ldots, \tag{27}$$

where L is the Lipschitzian constant from (6), c is the length of the edge of the first cube $\bar{K}_0 = \bar{C} \subset R^n$, $n = \dim C$ and $N \geqslant 2$ is the partition integer.

Deletion operator (24) excludes every subcube $\bar{C}_i^{j+1} \subset \bar{K}_j$ for which (24) is satisfied in its grid point $x_i^{j+1} \in \bar{C}_i^{j+1}$ and this, for each iteration $j = 0, 1, \ldots$. The closure of remaining subcubes constitutes the next set

$$\bar{K}_{j+1} = \{ x \mid x \in \bar{C}_i^{j+1}, i \in I_{j+1} \}, \qquad j = 0, 1, \ldots. \tag{28}$$

From this exclusion procedure it follows (see [1, inequality (4.11)]) that

$$\min_{x \in \bar{C}_i^{j+1}} f(x) > s_j, \qquad i \notin I_{j+1}, \quad j = 0, 1, \ldots \tag{29}$$

for every deleted subcube $\bar{C}_i^{j+1} \subset \bar{K}_j - K_{j+1}$, which implies

$$\min_{x \in \bar{Z}_{j+1}} f(x) > s_j, \qquad \bar{Z}_{j+1} = \bar{K}_j - K_{j+1}, \quad j = 0, 1, \ldots. \tag{30}$$

Superposition of (30) for $j=0, 1, \ldots, m$ yields in view of (7), (8),

$$\min_{x \in \bar{K}_0 - K_{m+1}} f(x) > s_m. \tag{31}$$

By virtue of (21), (22) we have

$$\bar{X} \subset \bar{K}_0 - K_{m+1} \tag{32}$$

so that, in view of (31), we have

$$p^0 = \min_{x \in \bar{X}} f(x) > s_m. \tag{33}$$

By the Liptschitz condition (6), we have for any subcube $\bar{C}_i^{m+1} \subset \bar{K}_m$,

$$\max_{x \in \bar{C}_i^{m+1}} f(x) - \min_{x \in \bar{C}_i^{m+1}} f(x) \leqslant L \max_{x,x' \in \bar{C}_i^{m+1}} \|x - x'\| = \frac{Lc\sqrt{n}}{N^{m+1}} = r_{m+1}. \tag{34}$$

After deletion by (24) for $j=m$, for all *remaining* subcubes $\bar{C}_i^{m+1} \subset \bar{K}_{m+1}$ we have

$$f(x_i^{m+1}) \leqslant s_m + r_{m+1}, \qquad x_i^{m+1} \in \bar{C}_i^{m+1} \subset \bar{K}_{m+1}. \tag{35}$$

Of course, (34), (35) are valid for any iteration $m = 0, 1, \ldots$.

Consider all subcubes $\bar{C}_i^{m+1} \subset \bar{Z}_{m+1} = \bar{K}_m - K_{m+1}$ such that $\bar{C}_i^{m+1} \cap \bar{X} \neq \varnothing$, they exist because of (21)–(23). By virtue of (29) for $j=m$, we have for each such subcube $\bar{C}_i^{m+1}$,

$$f(x)\,|_{x \in \bar{C}_i^{m+1} \cap \bar{X}} \geqslant \min_{x \in \bar{C}_i^{m+1}} f(x) > s_m. \tag{36}$$

On the other hand, we have for each $\bar{C}_i^{m+1}$,

$$f(x)\,|_{x \in \bar{C}_i^{m+1} \cap \bar{X}} \leqslant \max_{x \in \bar{C}_1^{m+1}} f(x) \leqslant \min_{x \in \bar{C}_i^{m+1}} f(x) + r_{m+1} \leqslant f(x_i^{m+1}) + r_{m+1}$$

$$\leqslant s_m + 2r_{m+1}, \tag{37}$$

where the second $\leqslant$ is due to (34) and the forth $\leqslant$ is by virtue of (35).

Since the global minimum

$$p^0 = \min_{x \in \bar{X}} f(x) \leqslant f(x)\,|_{x \in \bar{C}_i^{m+1} \cap \bar{X}}. \tag{38}$$

so, combining (33), (38), and (37), we obtain

$$s_m < p^0 \leqslant s_m + 2r_{m+1} = s_m + \frac{2Lc\sqrt{n}}{N^{m+1}}. \tag{39}$$

Now, combining (36) and (37) for the entire collection of $\bar{C}_i^{m+1} \subset \bar{Z}_{m+1} = \bar{K}_m - K_{m+1}$, we obtain

$$s_m + 2r_{m+1} \geqslant f(x)\,|_{x \in \bar{Z}_{m+1} \cap \bar{X}} > s_m. \tag{40}$$

Since (34), (35) are valid for any iteration, so (40) is also valid for all $m = 1, 2, \ldots$, such that there exist $\bar{C}_i^{m+1} \cap \bar{X} \neq \varnothing$, and we can rewrite it for some iteration l, $1 \leqslant l < m$,

$$s_l + 2r_{l+1} \geqslant f(x)\,|_{x \in \bar{Z}_{l+1} \cap \bar{X}} > s_l, \qquad \bar{Z}_{l+1} = \bar{K}_l - K_{l+1}, \quad \bar{Z}_{l+1} \cap \bar{X} \neq \varnothing. \tag{41}$$

Since $l < m$, so always $s_l \geqslant s_m$. If l is such that $s_l \leqslant s_m + 2r_{m+1}$, then the inequalities (40) and (41) are overlapping and we assert that $\bar{X}^0 \subseteq \bar{Z}_{m+1} = \bar{K}_m - K_{m+1}$. Suppose there exists such l, $1 \leqslant l \leqslant m-1$, that

$$s_l > s_m + 2r_{m+1} \geqslant s_{l+1}. \tag{42}$$

For this l we have the closest non-overlapping inequalities (40) and (41). Since $\bar{X}^0$ is the set of global minimizers $\{x^0\}$, $f(x^0) = p^0$ for $x^0 \in \bar{X}^0$, and since p^0 may be located anywhere within the semi-interval (39), so it follows that $\bar{X}^0 \cap \bar{Z}_{l+1} = \varnothing$ and, thus

$$\bar{X}^0 \subseteq \bigcup_{j=l+2}^{m+1} \bar{Z}_j = \bar{K}_{l+1} - K_{m+1} \qquad (0 \leqslant l \leqslant m-1) \tag{43}$$

with the understanding that for $l = 0$ we take only right inequality in (41).

Remark 3. If we assume again that (18) holds, then, since $r_m \to 0$, $s_m \to s^0$ as $m \to \infty$, we obtain (19) from (39). However, if (18) holds, then (22) does not, so the reasoning based on $\bar{Z}_m$ is invalid together with the result (43) (letting there $m \to \infty$ is incorrect and leads to the false statement: $\bar{X}^0 \subseteq \bar{K}_{l+1} - K^0$ since $\bar{K}_{m+1} \to \bar{K}^0$ as $m \to \infty$). Indeed, if (18) holds, then it may well happen that $\bar{Z}_{m+1} \cap \bar{X} = \varnothing$, so that (36), (37), (40), (41), and (42) disappear.

Also, if we *assumed* that $\bar{Z}_{m+1} \cap \bar{X} \neq \varnothing$ for all $m = 0, 1, \ldots$, then we would have $\lim_{m \to \infty} \bar{Z}_{m+1} = \lim_{m \to \infty} (\bar{K}_m - K_{m+1}) = \bar{K}^0 - K^0 = \partial K^0$ in which case we cannot pass to the limit in (40) as $m \to \infty$. We can, however, keep the argument (40)–(42) for each l (and m) such that $\bar{Z}_{l+1} \cap \bar{X} \neq \varnothing$ and without formally passing to the limit. Noting that $\bar{K}_{m+1} \cap \bar{X} \neq \varnothing$ for all m, see (18), we have to drop K_{m+1} from (43) altogether, and then, if there exist l, m such that (42) is satisfied, we obtain directly from (42) that $\bar{X}_0 \subseteq \bar{K}_{l+1}$ *for any finite* l *such that there exist* $m \geqslant l+1$ so that *for the pair* l, m *the inequalities* (42) *are satisfied.* This result is much weaker than (20) although not contradicting to it since from $\bar{X}^0 \subseteq \bar{K}^0$ and $\bar{K}^0 \subseteq \bar{K}_j$ for all j, see (7), (9), it follows $X^0 \subseteq \bar{K}_{l+1}$.

EXAMPLE 2. Consider the situation (2) when there exists $j_0 = m$ such that (21), (22) are satisfied and suppose that our first choice of the grid point $x_0 \in \bar{K}_0$ was so fortunate that we, in fact, hit the minimizing set: $x_0 \in \bar{K}^0 \subset \bar{K}_0 = \bar{C}$, *not knowing it in advance*. Then, to solve the problem, we still have to apply the cubic algorithm. In this case the comparison constant $s_0 = f(x_0)$ will remain in the process indefinitely because $s_0 = s^0$, due to $x_0 \in \bar{K}^0$, so that in (8) we shall have all equalities: $s_j = s_0 = \text{const.}$, $\forall j$. Then, there is no such l that the first inequality of (42) be satisfied, all inequalities (41) for $l = 0,..., m$ are overlapping with the same right-end value $s_l = s_0 = \text{const.}$, so that, instead of (43), we come to a rather poor evaluation

$$\bar{X}^0 \subseteq \bar{K}_1 - K_{m+1}, \tag{44}$$

which is trivial since from (5), $\bar{X} \subset \bar{C} = \bar{K}_0$ and (22), $\bar{K}_{m+1} \cap \bar{X} = \varnothing$ and (41), right inequality, the inclusion (44) follows. To improve the estimate (44) we have to introduce certain new devices into the algorithm.

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